Consumption and Savings Behaviour under Household Heterogeneity and Mortgage Debt

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CONSUMPTION AND SAVINGS BEHAVIOUR
UNDER HOUSEHOLD HETEROGENEITY AND MORTGAGE DEBT

Abstract

Since the GFC, mortgage debt and housing dynamics have been extensively researched. This issue is of central importance in understanding and analyzing individual consumption and savings behavior because housing takes up a large share in household illiquid assets, and most houses are financed with a mortgage loan.

To realistically capture the evolution of mortgage debt in the household income and wealth distribution, it is necessary to take into account the issue of heterogeneity. Households differ in levels of wealth and equity, and thus differ in their decisions to consume and save. The implications of this type of heterogeneity has been proven to have effects on aggregate consumption and output.

In this paper, I propose a model that incorporates i) heterogeneous households, ii) idiosyncratic labor income risk, and iii) incomplete markets. The model is set in a NK framework, enabling general equilibrium analysis. Furthermore, I utilize the finite difference method as an efficient method for solving for the general equilibrium. Some preliminary results indicate that under this setting, household consumption smoothing and precautionary savings motives still persist, and that the wealthy hand-to-mouth maintain a low level of consumption at all levels of mortgage indebtedness.

1. Introduction

The most recent Global Financial Crisis – also referred to as "the Great Recession" – has entailed some valuable lessons and insight for experts around the world. For years leading up to the Recession, the US housing market was experiencing a huge lending boom, which led to banks extending sub-prime mortgage loans in increasing amounts. The wake of the crisis was triggered by the burst of this housing bubble, where housing prices declined sharply, and homeowners found themselves suddenly facing negative equity.

Not surprisingly, in the years that followed the Great Recession, the issue of a mortgage lending boom and its subsequent effects on the overall economy became a topic vigorously discussed and studied by experts and policymakers around the world. While methods have varied across the board, there is little disagreement in the literature about the fact that housing takes up a large portion of household wealth, and its presence plays a vital role in the consumption and savings behavior of households. In other words, a person who has chosen to buy a house instead of renting will most certainly differ in consumption priorities, sensitivity, and reaction to shocks from someone who is abundant in cash and demand deposits. Thus, by taking into account this sort of asset allocation (from liquid to illiquid) in the household asset portfolio when setting up the household problem enables us to derive a more realistic representation of household consumption behavior.

Since the development of the real business cycle theory, the use of representative agent general equilibrium models with a New Keynesian framework have been the workhorse models used for macroeconomic analysis by policymakers. However, the recent crisis has also brought to light the fact that these models are becoming increasingly obsolete. Since then, much criticism has been directed towards the misgivings about models such as the Dynamic Stochastic General Equilibrium (DSGE) model, whose biggest downfall is perhaps the somewhat crude assumption of the existence of a representative agent in a given economy.

This assumption, while useful for computational analysis, diverges far from reality. It is difficult – if not impossible – to correctly identify and parametrize a model to fit a single household who can represent the whole economy, especially in a setting where income equality among households seems to be perpetually deteriorating. According to Inequality.org, the gap between the rich and the poor has been steadily and markedly widening for some 30 years now. A recent OECD report declared that the average Gini coefficient for household disposable income of its member countries had jumped from 0.315 in 2010 to 0.318 in 2014, which is a record high since the mid-1980's. Additionally, there are many more such data to indicate that income inequality is markedly increasing among households in many nations across the world.

It is mainly for these reasons that the development of recent literature has become focused on incorporating household heterogeneity into standard workhorse models. It has now become widely accepted that inequality and macroeconomics are closely correlated, and capturing idiosyncratic shocks and realistic wealth distributions will more than likely lead to a more realistic representation of consumption behavior and hence, improved policymaking.

Housing makes up a majority of household's wealth, and most houses are purchased with a mortgage loan, and this type of indebtedness renders the household's equity illiquid. Furthermore, having different levels of mortgage debt adds another level of heterogeneity to the household income and wealth distribution, and thus, the rationale for incorporating mortgage debt in the presence of heterogeneity is justified. Taking these issues into consideration, I build a heterogeneous agent model with liquid wealth and mortgage debt. The model is characterized by: i) household heterogeneity in holdings of liquid assets and subsidized mortgage debt, ii) idiosyncratic labor income shocks, and iii) incomplete markets.

To solve the model, I apply the upwind scheme finite difference method described in Achdou et al. (2017), and find that consumption smoothing and precautionary savings motives still play a significant role in household's behavior under the current setting. The rest of the paper is organized as follows: Section 3 proposes the model, and solves the problems of each agent individually. Section 4 defines the equilibrium and aggregates the model economy. In Section 5, I utilize the finite difference method to find the steady state approximation, and linearize the equilibrium conditions. Section 6 introduces some preliminary results describing the household consumption and savings behavior, and Section 7 concludes with a discussion.

2. Literature Review

The notion of heterogeneity is not a new concept in economic analysis, and has been developing since the introduction of Bewley (1986), Huggett (1993), and Aiyagari (1994) models. Since then, these models have become the standard workhorse models for conducting macroeconomic analysis in the presence of heterogeneity. The majority of heterogeneous agent models today are based upon the assumptions posed in these models: i) a continuum of ex ante heterogeneous agents with uninsured idiosyncratic income risks, and ii) an incomplete market characterized by the presence of borrowing constraints. These assumptions abstract from the Arrow Debreu representative agent economies, and argue that when they hold, the individual consumption and savings behavior varies significantly from that of complete market models.

On the other hand, Krussel and Smith (1998) have found that individual idiosyncratic shocks have an insignificant effect on the aggregate macroeconomic behavior, which can be fully characterized using only the mean of the wealth distribution. The model has later been revisited by Ahn et al. (2017), who applied a different approach to the computation of the model, and determined that income and wealth distribution do indeed affect the aggregate macroeconomic and welfare analysis.

Computational advances have made it possible to solve a wide array of heterogeneous agent models. The finite difference method upwind scheme (Barles and Souganidis, 1991) proposes a sophisticated and efficient method of solving heterogeneous agent models in continuous time (Achdou et al., 2017). Kaplan et al., (2016) utilize this method to build an extension of a standard workhorse heterogeneous agent model in a New Keynesian framework, and find that the presence of idiosyncratic shocks and a multiple-asset structure lead to significantly different responses to shocks in household behavior compared to the traditional representative model. In the Heterogeneous Agent New Keynesian (HANK) model setup, households' direct response to a one-time monetary shock is much less pronounced, and most of effects of the shock are transmitted through indirect channels.

Although a large amount of research has been conducted on the effects of mortgage debt on household's behavior, so far, most of them apply empirical methods, VAR analysis, or use a framework of representative agent models. Punzi et al. (2017) build a discrete-time TANK model to analyze the effects of an increase in housing investment risk on spender-saver households, and derive the conclusion that deleveraging effects on aggregate variables are significant and more pronounced than that of representative agent models. Hedlund et al. (2016) study the role of mortgage debt on the monetary policy transmission channels in a discrete-time HANK setup, and find that a substantial drop in consumption due to a contractionary monetary policy can be attributed to a decline in housing prices, and that monetary policy is more effective in a high LTV economy.

3. Heterogeneous Agent Model with Liquid Wealth and Mortgage Debt

The model incorporates mortgage debt into the standard New Keynesian economy with heterogeneous agents as in Kaplan et al. The main motivation behind this modification is to quantitatively analyze the effects of mortgage debt on household optimal consumption and savings policies under a general equilibrium. Time is continuous.

3.1 The Model Environment

The model economy is one populated by a continuum of households, who differ in their holdings of liquid assets b, level of mortgage debt m, and their idiosyncratic labor productivity z. Each household's lifetime spans infinity. On the production side, a final goods producer aggregates all goods produced by monopolistically competitive intermediate goods producers. Intermediate goods producers maximize their per-period profit, and are subject to price adjustment costs a la Rotemberg (1982). The introduction and role of financial intermediaries are inspired by Aoki and Nirei (2017). There is a large number of financial intermediaries who own the entire stock portfolio of all intermediary goods producers. These institutions internalize the risks of these illiquid assets, and sell riskless bonds to households in the form of liquid assets. The Government imposes a progressive tax on household labor income, and issues subsidized mortgage loans to households. There is a monetary authority who sets the nominal interest rate according to a Taylor rule.

3.1.1 Households

In each period, households make a decision to maximize their discounted lifetime utility, which is composed of preferences over consumption of non-durables, housing services, and labor. Thus, the objective function for households is given by:

$$E_0 \int_0^{\infty} e^{-\rho t} u(c_t, h_t, l_t) dt \tag{1}$$

Households are subjected to inelastic labor supply, meaning that they are willing to work the same amount of hours for a given wage rate w_t. Households receive utility flow from consumption and housing services, and disutility from hours worked. $\rho \geq 0$ is the discount rate for the future, and reflects the impatience of households.

Labor income is taxed proportionally, and is simultaneously subject to an idiosyncratic shock z_t. The labor productivity shock z follows a two-state Poisson process. Aside from taxing, the government also provides lump-sum transfers to households. $r_t^b$is the interest rate faced by households on their holdings of liquid wealth. I will define the flexible mortgage rate faced by households, r_t^m, so that it is directly tied to the nominal interest rate and the current period inflation rate:

$$r_t^m \equiv \beta i_t - \pi_t \tag{2}$$

Here, I also assume a linear relationship between the amount of housing services a household chooses and the mortgage debt amount m_t, and define $m_t \equiv \gamma_m h_t$. With this definition, it is implied that a household will have to take out a higher amount of payments if it wants to consume more housing services.

Because the economy that is being considered is one in which financial markets are underdeveloped, households are not presented with the opportunity to own productive forms of illiquid assets such as company shares and stocks. Therefore, in each given period, they can borrow and invest in a riskless bond b_t, and buy housing services (consequently resulting in a change in mortgage debt). When buying a house, individuals take out long-term, adjustable rate mortgage loans, and enter into an agreement to make interest rate payments in the amount of $r_t^m m_t$ from their current period income.

In addition to interest rate payments, households must pay off the mortgage debt through time. In each period, they can choose the optimal amount of debt repayment, d_t, by making portfolio reallocation from liquid wealth to mortgage debt repayment. This transaction is subject to an adjustment cost, $\kappa(d, m)$.

Households' liquid assets and mortgage debt stock evolve as follows:

$$\dot{b_t} = w_t l_t z_t - \widetilde{T}_t(w_t z_t l_t) + r_t^b b_t - d_t - \kappa(d, m) - c_t \tag{3}$$
$$\dot{m_t} = -r_t^m m_t - d_t \tag{4}$$

where $m \geq 0, \qquad b \geq \underline{b}$

In addition to these, households face a borrowing constraint on the amount of debt payable each period. I pose an assumption that the mortgage lender imposes some kind of restriction on the debt burden of borrowers, and that the relationship is linear with the income of that household.

$$r_t^m m_t \leq \phi w_t l_t \tag{5}$$

Households wealth and debt stock are also subject to the following transversality conditions designed for a no-Ponzi scheme requirement:

$$\lim_{t\to\infty} e^{-\int_0^t r_s^b ds} b(t) \geq 0 \tag{6}$$
$$\lim_{t\to\infty} e^{-\int_0^t r_s^m ds} m(t) = 0 \tag{7}$$

Households maximize (1) subject to (3), (4), (5), (6) and (7) to solve for optimal decisions for consumption, housing services, and labor hours, while taking time paths for taxes, transfers, real wages, real return to liquid assets, and the mortgage rate as given (determined in equilibrium). The household's problem is described recursively with a Hamilton-Jacobi-Bellman equation alongside the Kolmogorov Forward Equation in 3.2.1 of this Section.

3.1.2 Final Goods Producer

There is a competitive representative final goods producer who uses the CES aggregator to aggregate goods produced by intermediate producers. Goods are indexed by $j \in [0,1]$.

$$Y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{8}$$

3.1.3 Intermediate Goods Producers

In this New Keynesian framework, intermediate producers operate on a monopolistically competitive market, and produce according to a constant returns to scale technology using effective units of capital $k_{j,t}$ and effective units of labor $n_{j,t}$.

$$y_{j,t} = e^{Z_t} k_{j,t}^{\alpha} n_{j,t}^{1-\alpha} \tag{9}$$

where $Z_t = ln e^{Z_t}$ is the logarithm of aggregate productivity. The log productivity Z_t follows an Ornstein-Uhlenbeck process:

$$dZ_t = -\eta Z_t dt + \sigma dW_t \tag{10}$$

This is analogous to the AR(1) process in discrete time, in which W_t denotes a Wiener process that follows a standard Brownian motion, η is the rate of mean reversion, and σ captures the standard deviation of said process. We solve for intermediate firms' optimality conditions in Subsection 3.2.2.

3.1.4 Financial Intermediaries

In the model, I incorporate a large number of financial intermediaries with ownership of the entire stock portfolio of all intermediate goods producers. Their main objective is to convert risky stocks into risk-free bonds, which they then sell to households. This method of distributing monopoly profits of intermediate goods producers back to households was inspired by Aoki and Nirei (2017), and selected for the purpose of keeping the household asset portfolio free of additional illiquid assets.

3.1.5 Government

The active role of the government in this model is characterized by its aim to balance its budget, which is where its total expenditures are equal to its revenues from taxation. In this model, for the sake of simplicity, I abstract from government spending, and instead, assume that it is in charge of providing mortgage loans in the form of subsidies. Therefore, it receives all interest, net debt repayments and the respective transaction costs as revenue. Total mortgage debt stock in the economy is financed by the government; therefore, the change in total mortgage debt stock is the government's sole expenditure. Thus, the government balances its budget as below:

$$\dot{M}_t = \int \tilde{T}_t(w_t z l_t) d\mu_t + [D_t + \kappa(D, M)] + r^m M_t \tag{11}$$

Here I define progressive tax: $\tilde{T}_t(y) \equiv -T_t + \tau y$, where $T_t$is lump-sum transfers and τ denotes a proportional tax levied on household labor income y.

Due to heterogeneity in households, Ricardian equivalence breaks down. When a shock hits the economy, it can render an imbalance in the government's budget. Since I assume the government does not borrow in the bonds market, and the mortgage debt supply and repayment are strictly determined by households' decisions, it can rebalance its budget by adjusting the tax rate $\tilde{T}_t$ as necessary.

3.1.6 Monetary Authority

The central bank sets the nominal interest rate of the economy according to the Taylor rule:

$$i_t = \bar{r} + \phi \pi_t + u_t^{MP} \tag{12}$$

where $\bar{r}$ is the natural interest rate defined exogenously. u_t^{MP} is a monetary policy shock with mean zero, and follows the Ornstein-Uhlenbeck diffusion process:

$$dln(u^{MP}) = -\eta^u u^{MP} dt + \sigma^u dX_t \tag{13}$$

where dX_t is an innovation to the standard Brownian motion, η^u is its mean reversion rate, and σ^u captures the size of the innovation. The real liquid interest rate, r_t^b, is derived from the Fischer equation as follows:

$$r_t^b = i_t - \pi_t \tag{14}$$

3.2 Solutions

In this Sub-section, I will solve the problems faced by agents in the economy separately for the purpose of deriving their optimal decision rules. Subsection 3.2.1 solves the households' utility maximization problem using the Hamilton-Jacobi-Bellman (HJB) and derives the joint distribution of wealth and mortgage debt using the Kolmogorov Forward Equation (KFE) or the Fokker-Planck equation. Subsection 3.2.2 solves the firms' profit maximization problem for the optimality conditions along the line of a New Keynesian framework, and 3.2.3 presents the profit maximization problem for the financial intermediaries.

3.2.1 Households' Problem

The households' HJB equation is given as follows:

$$\begin{aligned}\rho V(b,m,z) = \max_{c_t,h_t,l_t} & \; u(c,h,l) + V_b(b,m,z)(w_t z_t l_t - \tilde{T}_t + r^b b_t - d_t - \kappa(d,m) - c \\ & + V_m(-r^m m_t - d_t) \\ & + \sum \lambda(z_1,z_2)\{(V(b,m\, z_2) - V(b,m,z_1))\} + \frac{1}{dt}E_t[dV(b,m,z)]\end{aligned} \tag{15}$$

To derive the optimal household decisions, we must analyze the first order conditions:

1. FOC w.r.t c_t:

$$u_c(c,h,l) = V_b(b,m,z) \tag{16}$$

2. FOC w.r.t h_t:

$$u_h(c,h,t) = \gamma_h r^m V_m(b,m,z) \tag{17}$$

$$\because m \equiv \gamma_h h_t \Rightarrow \dot{m} = -r^m \gamma_h h_t - d_t$$

3. FOC w.r.t l_t:

$$u_l(c,h,l) = (-wz + \tilde{T})V_b(b,m,z) \tag{18}$$

4. FOC w.r.t d_t:

$$[-1 - \kappa_d(d,m)]V_b(b,m,z) = V_m(b,m,z) \tag{19}$$

Before commencing any further with solving for optimal decisions corresponding to the household's HJB, let us provide a couple of additional details on the model that are related to this part.

A. CRRA utility function

The model assumes that each household has time separable preferences and instantaneous utility over consumption, housing, and labor. For analytical simplicity, the utility function takes the functional form of constant relative risk aversion (CRRA).

$$u(c,h,l) = \frac{1}{1-\gamma}c^{1-\gamma} + \frac{1}{1-\varphi}h^{1-\varphi} - \frac{1}{1-\psi}l^{1-\psi} \tag{20}$$

B. Adjustment Cost Function

As mentioned previously, mortgage refinancing/repayment is carried out in a similar fashion to portfolio reallocation, in the sense that the payment is made out from liquid earnings. Thus, I adopt a similar adjustment cost function to that of Kaplan et al. (2016). This adjustment cost is a function of debt repayment/refinancing and mortgage debt stock, and takes the following form:

$$\kappa(d,m) = \kappa_0|d| + \frac{\kappa_1}{2}\left(\frac{d}{\max\{m,\underline{m}\}}\right)^2 \max\{m,\underline{m}\},\ \underline{m} > 0 \tag{21}$$

This functional form is assumed for the sake of having a linear part, which represents an inaction region, and a convex part that which ensures the finite nature of repayment/refinancing rate ($|d_t| < \infty$).

Now that the functional forms for the household's preferences and the adjustment costs have been defined, we can derive the optimal decision rules for consumption, savings, housing, and mortgage repayment rate. To do so, we can utilize equations (16) - (19) derived from the HJB equation.

1. Consumption:

$$c^{-\gamma} = V_b(b,m,z)$$

$$\Rightarrow \quad c^*(b,m,z) = V_b(b,m,z)^{-\frac{1}{\gamma}} \tag{22}$$

2. Housing:

$$h^{-\varphi} = \gamma_h r_t^m V_m(b,m,z)$$

$$\Rightarrow \quad h^*(b,m,z) = [\gamma_h r_t^m V_m(b,m,z)]^{-\frac{1}{\varphi}} \tag{23}$$

3. Labor:

$$l^{-\psi} = \left(-wz + \tilde{T}_t\right) V_b(b,m,z)$$

$$\Rightarrow \quad l^* = \left[\left(-wz + \tilde{T}\right) V_b(b,m,z)\right]^{-\frac{1}{\psi}} \tag{24}$$

4. Repayment:

Given that:

$$\kappa_d(d,m) = \begin{cases} \kappa_0 + \kappa_1 \dfrac{d}{m}, & d > 0 \\ -\kappa_0 + \kappa_1 \dfrac{d}{m}, & d < 0 \end{cases}$$

Substituting into FOC and rearranging:

$$\frac{\kappa_1}{m} d = \begin{cases} -\dfrac{V_m(b,m,z)}{V_b(b,m,z)} - 1 - \kappa_0, & d > 0 \\ -\dfrac{V_m(b,m,z)}{V_b(b,m,z)} - 1 + \kappa_0, & d < 0 \end{cases}$$

$$\Rightarrow \quad d^* = \begin{cases} \left(-\dfrac{V_m(b,m,z)}{V_b(b,m,z)} - 1 - \kappa_0\right) \dfrac{m}{\kappa_1}, & d > 0 \\ \left(-\dfrac{V_m(b,m,z)}{V_b(b,m,z)} - 1 + \kappa_0\right) \dfrac{m}{\kappa_1}, & d < 0 \end{cases}$$

In general form, we can rewrite the equation for optimal repayment as follows:

$$d^* = \left(-\frac{V_m(b,m,z)}{V_b(b,m,z)} - 1 + \kappa_0\right)^- \frac{m}{\kappa_1} + \left(-\frac{V_m(b,m,z)}{V_b(b,m,z)} - 1 - \kappa_0\right)^+ \frac{m}{\kappa_1} \tag{25}$$

5. Liquid savings:

$$s_b^*(b,m,z) = w_t z l_t - \tilde{T} + r_t^b b_t - d_t - \kappa_0 |d| + \frac{\kappa_1}{2}\left(\frac{d}{m}\right)^2 m - c \tag{26}$$

6. Mortgage debt build-up:

$$s_m^*(b,m,z) = -r_t^m m_t - d_t \tag{27}$$

The Kolmogorov Forward Equation (Fokker-Planck Equation) provides the joint distributions of households' liquid assets (b), debt (m), and idiosyncratic labor productivity (z). This distribution is essentially a density function, and is denoted as $g(b,m,z,t)$. Also, the optimal liquid asset savings policy function and the optimal mortgage debt build-up policy functions in (26) and (27) can be interpreted as the optimal drifts in the HJB equation.

$$\frac{dg_t(b,m,z)}{dt} = -\partial_b[s_t^b(b,m,z)g_t(b,m,z)] - \partial_m[s_t^m(b,m,z)g_t(b,m,z)] - \lambda_1 g_t(b,m,z_1) + \\ +\lambda_2 g_t(b,m,z_2) \tag{28}$$

The household's consumption and savings policy functions are derived from the HJB equation, and the joint distribution of individual state variables b, m, z is described by the KFE. Solving these two equations will yield the solution to the household's problem. In actuality, there is a numerical solution to the HJB equation, which is a "constrained viscosity solution", and using the upwind finite difference method as in Achdou et al. (2017), the solution to the HJB essentially gives the solution to the KFE "for free". A more detailed solution using the finite difference method is provided in Appendix I.

3.2.2 Final Goods Producer's Problem

The representative final goods producer aggregates j types of intermediate goods according to (8). Here, the elasticity of substitution across goods is given by $\varepsilon > 0$. The profit maximization problem of the final producer is characterized as follows:

$$\max_{y_{j,t}} P_t \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} \, dj \right)^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^1 p_{j,t} \, y_{j,t} dj \tag{29}$$

Then, the FOCs for a typical intermediate good *j* is as follows:

$$P_t \frac{\varepsilon}{\varepsilon-1} \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} \, dj \right)^{\frac{\epsilon}{\varepsilon-1}-1} \frac{\varepsilon-1}{\varepsilon} y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}-1} = p_{j,t} \tag{30}$$

This can be written as:

$$\left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} \, dj \right)^{\frac{1}{\varepsilon-1}} y_{j,t}^{-\frac{1}{\varepsilon}} = \frac{p_{j,t}}{P_t}$$

or:

$$\left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} \, dj \right)^{-\frac{\varepsilon}{\varepsilon-1}} y_{j,t}^{-\frac{1}{\varepsilon}} = \left(\frac{p_{j,t}}{P_t} \right)^{-\varepsilon} \tag{31}$$

Utilizing the definition of the aggregate final good, we derive the demand for intermediate good *j:*

$$y_{j,t}(p_{j,t}) = \left(\frac{p_{j,t}}{P_t} \right)^{-\varepsilon} Y_t, \tag{32}$$

The downward sloping demand curve for intermediate good *j* in (32) implies that the relative demand for this good is a function of its relative price, and is proportional to aggregate output Y_t. Here, ε is the price elasticity of demand.

In order to derive a price index for the aggregate price level, we apply the definition of nominal output as the sum of prices multiplied by quantities.

$$P_t Y_t = \int_0^1 p_{j,t} y_{j,t} dj$$

Plugging in demand for *j*-th good derived in (32) into the above definition:

$$P_t Y_t = \int_0^1 p_{j,t}^{1-\varepsilon} P_t^{\varepsilon} Y_t dj$$

$$\Rightarrow P_t Y_t = P_t^{\varepsilon} Y_t \int_0^1 p_{j,t}^{1-\varepsilon} dj$$

Simplify above equation, we obtain the price index:

$$P_t = \left(\int_0^1 p_{j,t}^{1-\varepsilon}\, dj\right)^{\frac{1}{1-\varepsilon}} \tag{33}$$

3.2.3 Intermediate Firm's Problems

The monopolistically competitive intermediate firms produce differentiated goods indexed $j = [0, 1]$ according to the production technology in (11), and subject to aggregate productivity shock in (12).

Solving the cost minimization problem results in factor prices faced by an intermediate producer equaling their respective marginal revenue products, and the marginal costs are given as follows:

$$mc_t = \frac{1}{e^Z}\left(\frac{r_t^k}{\alpha}\right)^{\alpha}\left(\frac{w_t}{1-\alpha}\right)^{1-\alpha} \tag{34}$$

Here, r_t^k is the rate of rent of capital, and $w_t$is real wages to hire the labor to produce intermediate goods.

The model assumes that in equilibrium, there is no inflation, i.e., $\frac{p_{j,t}}{P_t} = 1$. Aggregate amount of labor and capital used by firms will be equal to the sum of all inputs used by all firms, and that factor prices are determined on a competitive market. In this case, we can simply derive factor prices as follows:

$$r_t^k = \alpha e^Z K^{\alpha-1} N^{1-\alpha} - \delta \tag{35}$$

$$w_t = (1-\alpha) e^Z K^{\alpha} N^{-\alpha} \tag{36}$$

Due to the monopolistically competitive nature of intermediate producers, each firm faces a downward sloping demand curve as in (32), which means that it has monopoly power to set its price. Intermediate firms choose their prices to maximize their respective profits, but face quadratic price adjustment costs a la Rotemberg:

$$\Theta_t\left(\frac{\dot{p}_t}{p_t}\right) = \frac{\theta}{2}\left(\frac{\dot{p}_t}{p_t}\right)^2 Y_t = \frac{\theta}{2}(\pi)^2 Y_t \tag{37}$$

Taking the price adjustment cost in (37) into account, the profit maximization problem of an intermediate goods producer is given by:

$$\int_0^\infty e^{-\int_0^t r_s^b ds}\left\{\Pi_t(p_t) - \Theta_t\left(\frac{\dot{p}_t}{p_t}\right)\right\} dt \tag{38}$$

where per-period profits are given by:

$$\Pi_t(p_t) = \left(\frac{p_t}{P_t} - mc_t\right)\left(\frac{p_t}{P_t}\right)^{-\varepsilon} Y_t \tag{39}$$

Now, I will utilize the recursive form of the profit maximization problem of the intermediate firm to derive the Phillips Curve in this model. The treatment here largely follows that of Kaplan et al (2016). Here, I denote the real value of a firm with price p as $Q(p, t)$.

$$r_t^b Q(p,t) = \max_{\pi}\left(\frac{p}{P_t} - mc_t\right)\left(\frac{p}{P_t}\right)^{-\varepsilon} Y_t - \frac{\theta}{2}\pi^2 Y_t + Q_p(p,t)p\pi + Q_t(p,t) \tag{40}$$

Then, the first order and envelope conditions are derived as follows:

$$Q_p(p,t)p = \theta\pi Y$$

$$(r^b - \pi)Q_p(p,t) = -\left(\frac{p}{P} - mc\right)\varepsilon\left(\frac{p}{P}\right)^{-\varepsilon-1}\frac{Y}{P} + \left(\frac{p}{P}\right)^{-\varepsilon}\frac{Y}{P} + Q_{pp}(p,t)p\pi + Q_{tp}(p,t)$$

Again, in equilibrium, it is assumed that $p = P$. Therefore, the set of equations above will collapse to:

$$Q_p(p,t) = \frac{\theta\pi Y}{p} \tag{41}$$

$$\left(r_t^b - \pi\right)Q_p(p,t) = -(1 - mc)\varepsilon\frac{Y}{p} + \frac{Y}{p} + Q_{pp}(p,t)p\pi + Q_{tp}(p,t) \tag{42}$$

Differentiating (41) with respect to time will yield:

$$Q_{pp}(p,t)\dot{p} + Q_{pt}(p,t) = \frac{\theta Y \dot{\pi}}{p} + \frac{\theta \dot{Y} \pi}{p} - \frac{\theta Y}{p}\frac{\dot{p}}{p}$$

Plug this result into equation (42), and divide by $\theta Y/p$ to derive:

$$\left(r^b - \frac{\dot{Y}}{Y}\right)\pi = \frac{1}{\theta}[-(1-mc)\varepsilon + 1] + \dot{\pi}$$

Rearrange the variables to attain the New Keynesian Phillips curve in continuous time:

$$\left(r_t^b - \frac{\dot{Y}_t}{Y_t}\right)\pi_t = \frac{\varepsilon}{\theta}(mc_t - mc^*) + \dot{\pi}_t \tag{43}$$

We can write the equation in (43) in present-value form:

$$\pi_t = \frac{\varepsilon}{\theta}\int_t^{\infty} e^{-\int_t^s r_\tau^b d\tau} \frac{Y_s}{Y_t}(mc_s - mc^*)ds \tag{44}$$

Here, $\mathcal{M}^* = \frac{1}{mc^*} = \frac{\varepsilon}{\varepsilon - 1}$ is the flexible price optimum markup, while $\mathcal{M}_s = \frac{1}{mc_s}$ is a firm's markup at time s. Intermediate firms will choose to increase their price if $\mathcal{M}_s < \mathcal{M}^*$.

3.2.4 Financial Intermediary's Problem

There is a large number of perfectly competitive, identical financial intermediaries whose sole purpose is to redistribute firms' profits to households. For the purpose of simplicity, I assume that each firm's net worth is equal to zero. In the process of redistributing firms' profits, they receive revenue flow from their shares of intermediate firms, and incur costs from interest paid on risk-free, liquid bonds that they issue to households. An additional assumption here is that the financial intermediary also rents out physical capital to intermediate firms through a competitive rental market. Therefore, in equilibrium, the financial intermediaries' income from ownership of firms is equal to intermediate producers' profits:

$$q_t K_t = (1 - mc)Y \tag{45}$$

where q_t is dividend rate per unit of capital, and $(1-m)Y$ is firms' profits in given by equation (39) in the case of no inflation.

Finally, the financial intermediaries earn income on capital rent, and pay interest on bonds issued. This, in simple form, their per-period profits are given as below:

$$\Pi_t^{fi} = q_t K_t + \left(r_t^k - \delta\right) K_t - r_t^B B_t^f \tag{46}$$

Let us denote the net worth of a financial intermediary as $W = K - B^f$. The equilibrium condition is that $W = 0$, and hence $K = B^f$. Therefore, we can substitute this condition into (46) to attain:

$$\Pi_t^{fi} = q_t K_t + \left(r_t^k - \delta\right) K_t - r_t^B K_t$$

Maximize profits by taking FOC w.r.t K_t and equating to zero, we get:

$$r_t^b = q_t + r_t^k - \delta \tag{47}$$

4 Equilibrium and Aggregation

In this Section, we define and derive the aggregate equilibrium to close the model economy.

4.1 Definition of Equilibrium

An equilibrium in this economy is defined as price paths for prices $\{w_t, r_t^k, r_t^b, r_t^m\}$, and corresponding allocations, and a government policy path for taxation $\{\widetilde{T}_t\}$, such that:

(i) Households, intermediate goods producers, and financial intermediaries maximize their objective functions, taking as given equilibrium prices $\{w_t, r_t^k, r_t^b, r_t^m\}$ and taxation policy $\{\widetilde{T}_t\}$,

(ii) The government budget constraint holds, and

(iii) All four markets clear as follows:

a. The liquid asset market clears when total household saving in risk-free bonds equals bonds issued by the financial intermediaries:

$$B_t^h = B_t^f$$

b. The capital market clears when the total amount of capital used in production is in equilibrium with household saving in liquid assets issued through the financial intermediaries:

$$K_t = B_t^f$$

c. The labor market clears when:

$$N_t = \int z l_t(b, m, z) dbdmdz$$

d. The goods market clearing condition is given as follows:

$$Y_t = C_t + I_t + \kappa(D, M)$$

4.2 Aggregation

Given the definition of equilibrium in the previous Subsection, I will now make a summary of all aggregate variables of the economy in this state to ensure that the model is closed. We have 26 variables $\{r^b, r^m, w, r^k, \Pi, mc, Y, K, q, B^f, B^h, i, \pi, mc^*, I, C, \kappa, D, M, \tilde{T}, \dot{M}, H, N, c, h, l\}$, and the corresponding 26 equations to close the model.

$$r_t^k = \alpha e^Z K^{\alpha-1} N^{1-\alpha} - \delta \quad (48)$$

$$w_t = (1-\alpha) e^Z K^{\alpha} N^{-\alpha} \quad (49)$$

$$\Pi = (1 - mc)Y \quad (50)$$

$$mc = \frac{1}{e^Z} \left(\frac{r^k}{\alpha}\right)^{\alpha} \left(\frac{w}{1-\alpha}\right)^{1-\alpha} \quad (51)$$

$$Y = \frac{qK}{1-mc} \quad (52)$$

$$q = r^b - (r^k - \delta) \quad (53)$$

$$r^b = i - \pi \quad (54)$$

$$i = \phi\pi + \bar{r} \quad (55)$$

$$\pi = \frac{\varepsilon}{\theta} \int_t^{\infty} e^{-\int_t^s r_\tau^b d\tau} \frac{Y_s}{Y_t} (mc_s - mc^*) ds \quad (56)$$

$$mc^* = \frac{\varepsilon-1}{\varepsilon} \tag{57}$$

$$K = B^f \tag{58}$$

$$B^f = B^h \tag{59}$$

$$B^h = \int_{\underline{b}}^{\infty} bg_1(b,m,z)dbdmdz + \int_{\underline{b}}^{\infty} bg_2(b,m,z)dbdmdz \tag{60}$$

$$r^m = \beta i - \pi \tag{61}$$

$$I = Y - C - \kappa(D,M) \tag{62}$$

$$C = \int c^*(b,m,z)dbdmdz \tag{63}$$

$$\kappa(D,M) = \kappa_0|D| + \frac{\kappa_1}{2}\left(\frac{D}{M}\right)^2 M \tag{64}$$

$$D = \int d^*(b,m,z)dbdmdz \tag{65}$$

$$M = \int_0^{\infty} mg_1(b,m,z_1)dbdmdz + \int_0^{\infty} mg_2(b,m,z_2)dbdmdz \tag{66}$$

$$\tilde{T} = \dot{M} - D - \kappa(D,M) - r^m M \tag{67}$$

$$\dot{M} = -r^m M - D \tag{68}$$

$$H = \int h^*(b,m,z)dbdmdz \tag{69}$$

$$N = \int zl^*(b,m,z)dbdmdz \tag{70}$$

$$c^*(b,m,z) = V_b(b,m,z)^{-\frac{1}{\gamma}} \tag{71}$$

$$h^*(b,m,z) = [\gamma_h r_t^m V_m(b,m,z)]^{-\frac{1}{\varphi}} \tag{72}$$

$$l^* = \left[\left(-wz + \tilde{T}\right)V_b(b,m,z)\right]^{-\frac{1}{\psi}} \tag{73}$$

Equations (48) - (73) aggregate the model in equilibrium. However, although we can pin down all 26 variables, due to the rich heterogeneity in households' states and behaviors in this economy, equations related to this block of the model cannot be solved analytically. Thus, I propose an alternative solution to this model in Section 5.

5. Alternative Solution Method

The alternative solution method I apply to this model is widely used in models with rich heterogeneity. The method involves using the finite difference method as in Achdou et al. (2017) to approximate variables, then discretizing and linearizing the model around the steady state to make it tractable. The finite difference method an efficient and robust method for solving heterogeneous agent models in continuous time, and was used by Kaplan et al. (2017) to solve the Krussel and Smith (1998) model. The treatment here closely follows that of Kaplan et al. (2017). The finite difference method applied to derive the approximations is explained in Appendix I of this paper.

Before commencing any further, let me simplify the model for computational convenience. The simplifying assumptions essentially involve setting labor supply exogenously, and excluding the activities of the government and the central bank. Thus, an equilibrium of the model can now we simply characterized as follows:

$$\rho V(b,m,z) = \max_{c} u(c_t) + V_b(b,m,z)\big(w_t z_t + r_t^b b_t - d_t - \kappa(d,m) - c_t\big) + V_m(-r_t^m m_t - d_t + \sum \lambda(z_1,z_2)[V(b,m,z_2) - V(b,m,z_1)] + \frac{1}{dt}E_t[dV(b,m,z)] \tag{74}$$

$$\frac{dg_t(b,m,z)}{dt} = -\partial_b[s_t^b(b,m,z)g_t(b,m,z)] - \partial_m[s_t^m(b,m,z)g_t(b,m,z)] - \lambda g_t(b,m,z_1) + \lambda g_t(b,m,z_2) \tag{75}$$

$$dZ_t = -\eta Z_t dt + \sigma dW_t \tag{76}$$

$$w_t = (1-\alpha)e^{Z_t}K_t^{\alpha}\bar{N}^{-\alpha} \tag{77}$$

$$r_t^b = \alpha e^{Z_t}K_t^{\alpha-1}\bar{N}^{1-\alpha} - \delta \tag{78}$$

$$r_t^m = \beta r_t^b \tag{79}$$

$$K_t = \int b g_t(b,m,z)dbdmdz \tag{80}$$

$$M_t = \int m g_t(b,m,z)dbdmdz \tag{81}$$

where $s_t^b(b,m,z) = w_t z + r_t^b b_t - d_t - \kappa(d,m) - c_t$ and $s_t^m(b,m,z) = -r_t^m m_t - d_t$ are the optimal savings and debt accumulation policies.

The HJB and KFE equations in this part are identical to those described in the household's problem in Subsection 3.2.1, with a minor modification of the exclusion of taxes and exogenous labor supply for simplicity. Note that these can all be added to the method applied here without loss of efficiency.

Now that the model is simplified and closed, the next logical step is to define a steady state. The steady state in this case is defined as an equilibrium where aggregate productivity is constant at $Z_t = 0$, and the joint distribution of liquid wealth, mortgage debt and idiosyncratic shock $g(b, m, z)$ is time-invariant. Hence, the steady state can be defined by the following set of equations:

$$\rho V(b,m,z) = \max_c u(c) + V_b(b,m,z)(wz + r^b b - d - \kappa(d,m) - c) + V_m(-r^m m - d$$
$$+ \sum \lambda(z_1, z_2)[V(b,m,z_2) - V(b,m,z_1)], \quad m < wz \tag{82}$$

$$0 = -\partial_b[s^b(b,m,z)g(b,m,z)] - \partial_m[s^m(b,m,z)g(b,m,z)] - \lambda g(b,m,z_1) + \lambda g(b,m,z_2) \tag{83}$$

$$w = (1-\alpha)K^\alpha \bar{N}^{-\alpha} \tag{84}$$

$$r^b = \alpha K^{\alpha-1}\bar{N}^{1-\alpha} - \delta \tag{85}$$

$$r^m = \beta r^b \tag{86}$$

$$K = \int bg(b,m,z)dbdmdz \tag{87}$$

$$M = \int mg(b,m,z)dbdmdz \tag{88}$$

Now that the steady state without aggregate shocks is defined, the next step is to approximate the steady state using a linearization procedure. The approximation method used here is the finite difference methods described in Achdou et al. (2017). The value function and distribution are approximated over discretized grids of liquid asset holdings $\boldsymbol{b} = (b_1 = 0, b_2, \dots, b_I)^T$ and mortgage debt $\boldsymbol{m} = (m_1 = 0, m_2, \dots, m_I)^T$. The value function over these grids is denoted by:

$$\boldsymbol{v} = \begin{pmatrix} v(b_1, m_1, z_1) & \dots & v(b_I, m_1, z_2) \\ v(b_1, m_1, z_1) & \dots & v(b_1, m_I, z_2) \end{pmatrix}^T$$

and the distribution over the grids is given by:

$$\boldsymbol{g} = \begin{pmatrix} g(b_1, m_1, z_1) & \dots & g(b_I, m_1, z_2) \\ g(b_1, m_1, z_1) & \dots & g(b_1, m_I, z_2) \end{pmatrix}^T$$

The dimension for both $\boldsymbol{v}$ and $\boldsymbol{g}$ is $N \times 2$, where $N = 2I$. Equations (82) and (83) are approximated at each point on the grids using an “upwind” scheme to approximate the partial derivatives as in Appendix I. Then, the approximated steady state collapses into the following set of equations:

$$\rho \boldsymbol{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v};\mathbf{p})\mathbf{v} \tag{89}$$

$$\mathbf{0} = \boldsymbol{A}(\boldsymbol{v};\boldsymbol{p})^T \boldsymbol{g} \tag{90}$$

$$\boldsymbol{p} = \boldsymbol{F}(\boldsymbol{g}) \tag{91}$$

where $\mathbf{u}(\mathbf{v})$ is a matrix of the maximized utility function over the grids, and matrix $\mathbf{A}(\mathbf{v};\mathbf{p})\mathbf{v}$ captures the remaining terms in (82), including the deposit rate and adjustment cost function as expressed by d and m. Equation (90) is the discretized version of the KFE described in (83), and (91) describes the movement of prices $\boldsymbol{p} = (r^b, w)^T$ as a function of aggregate capital and mortgage debt along the joint distribution $\boldsymbol{g}$. Note that we can forget about the mortgage rate since it does not depend on the movement of capital, but is rather tied to the liquid rate by an exogenous parameter β. The steady state system is solvable because, in total, there are $4N + 2$ equations and $4N + 2$ unknowns.

Having approximated the steady state, it is now possible to linearize the discretized equilibrium conditions stated in (74) - (81). The discretized equilibrium can be fully described by the following system of equations, which consists of $4N + 3$ stochastic differential equations in $4N + 3$ unknowns:

$$\rho \boldsymbol{v_t} = \mathbf{u}(\mathbf{v_t}) + \mathbf{A}(\mathbf{v_t};\mathbf{p_t})\mathbf{v_t} + \frac{1}{dt} E_t d\boldsymbol{v_t} \tag{92}$$

$$\frac{d\boldsymbol{g_t}}{dt} = \boldsymbol{A}(\boldsymbol{v_t};\boldsymbol{p_t})^T \boldsymbol{g_t} \tag{93}$$

$$dZ_t = -\eta Z_t dt + \sigma dW_t \tag{94}$$

$$\boldsymbol{p_t} = \boldsymbol{F}(\boldsymbol{g_t}; Z_t) \tag{95}$$

As an intermediate step, I shall rearrange the equations so that all time derivatives are aligned on the left. Take an expectation of the entire system, and note that the expectation of a Wiener process is equal to zero.

$$E_t \begin{bmatrix} d\mathbf{v}_t \\ d\mathbf{g}_t \\ dZ_t \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{u}(\mathbf{v}_t;\mathbf{p}_t) + \mathbf{A}(\mathbf{v}_t;\mathbf{p}_t)\mathbf{v}_t - \rho\mathbf{v_t} \\ \mathbf{A}(\mathbf{v}_t;\mathbf{p}_t)^T\mathbf{g}_t \\ -\eta Z_t \\ \mathbf{F}(\mathbf{g_t};Z_t) - \mathbf{p_t} \end{bmatrix} dt$$

Then, the first-order Taylor expansion is as follows:

$$E_t \begin{bmatrix} d\hat{\mathbf{v}}_t \\ d\hat{\mathbf{g}}_t \\ dZ_t \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{vv} & 0 & 0 & \mathbf{B}_{vp} \\ \mathbf{B}_{gv} & \mathbf{B}_{gg} & 0 & \mathbf{B}_{gp} \\ 0 & 0 & -\eta & 0 \\ 0 & \mathbf{B}_{pg} & \mathbf{B}_{pZ} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{v}}_t \\ \hat{\mathbf{g}}_t \\ Z_t \\ \hat{\mathbf{p}}_t \end{bmatrix} dt \tag{96}$$

where $\widehat{\boldsymbol{v}}_{\boldsymbol{t}}, \widehat{\boldsymbol{g}}_{\boldsymbol{t}}, Z_t, \widehat{\boldsymbol{p}}_{\boldsymbol{t}}$ are expressions of the value function, distribution, aggregate productivity, and prices in terms of deviations from the steady state. Notice that the pricing equation is static, with zero expectation of derivative with respect to time, therefore allowing us to utilize the simplifying condition $\widehat{\boldsymbol{p}}_t = \boldsymbol{B}_{pg}\widehat{\boldsymbol{g}}_t + \boldsymbol{B}_{pZ}Z_t$ to substitute into (96).

$$E_t \begin{bmatrix} d\widehat{\boldsymbol{v}}_t \\ d\widehat{\boldsymbol{g}}_t \\ dZ_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_{vv} & \boldsymbol{B}_{vp}\boldsymbol{B}_{pg} & \boldsymbol{B}_{vp}\boldsymbol{B}_{pZ} \\ \boldsymbol{B}_{gv} & \boldsymbol{B}_{gg} + \boldsymbol{B}_{gp}\boldsymbol{B}_{pg} & \boldsymbol{B}_{gp}\boldsymbol{B}_{pZ} \\ 0 & 0 & -\eta \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{v}}_t \\ \widehat{\boldsymbol{g}}_t \\ Z_t \end{bmatrix} dt \tag{97}$$

If the Blanchard-Kahn condition holds, the system can be solved as below:

$$\widehat{\boldsymbol{v}}_t = \boldsymbol{D}_{vg}\hat{g}_t + \boldsymbol{D}_{vZ}Z_t \tag{98}$$

$$\frac{d\widehat{\boldsymbol{g}}_t}{dt} = \left(\boldsymbol{B}_{gg} + \boldsymbol{B}_{gp}\boldsymbol{B}_{pg} + \boldsymbol{B}_{gv}\boldsymbol{D}_{vg}\right)\widehat{\boldsymbol{g}}_t + \left(\boldsymbol{B}_{gp}\boldsymbol{B}_{pZ} + \boldsymbol{B}_{gv}\boldsymbol{D}_{vZ}\right)Z_t \tag{99}$$

$$dZ_t = -\eta Z_t dt + \sigma dW_t \tag{100}$$

$$\widehat{\boldsymbol{p}}_t = \boldsymbol{B}_{pg}\widehat{\boldsymbol{g}}_t + \boldsymbol{B}_{pZ}Z_t \tag{101}$$

where $\boldsymbol{D}_{vg}$ and $\boldsymbol{D}_{vZ}$ are defined in this context as household's optimal policy rules under the effects of aggregate shocks. After observing the system given by (98) – (101), one can argue that the solution to the heterogeneous agent model is reduced to a structure similar to that of standard representative agent models such as the Real Business Cycle (RBC) model. This result is also significant in that it proves that the finite difference method can be applied to heterogeneous agent models with different structures of household assets.

6. Preliminary Results: Household Consumption and Savings Behavior

In this Section, I have aimed to discuss some preliminary results derived from experimental computation exercises performed on this framework. The reader should be noted that these results are not final, but are enough to demonstrate some behaviors of the households with rich heterogeneity. As this class of models is still relatively new, and although much progress is made in terms of computation of household decisions, it still poses a challenge in terms of computation of a general equilibrium under the given settings.

For this exercise, I take the block of the model which features household decisions to analyze the optimal consumption, savings and debt accumulation decisions of households. As described in Subsection 3.2.1, the HJB equation and KFE are set up. Then, following the same solution method, the first order conditions were derived according to (22) - (27). The finite difference method was used to approximate the partial derivatives of the value function and the joint distribution (Please refer to Appendix I for more details on the upwind finite difference scheme used for this problem).

As we can see from the results of this exercise, idiosyncratic labor productivity plays a significant role in determining the consumption and savings decisions of households. In Figure 1, we observe consumption-smoothing behavior in both cases. However, household consumption rises much faster as liquid wealth increases. Interestingly, there are some households who have zero liquid wealth and are highly indebted (i.e., "hand-to-mouth wealthy households" who own a house but are cash-constrained) maintain a low level of consumption throughout in both states. This reflects that when hit by an aggregate shock, the consumption response of the majority of these households will be higher in magnitude. It is also worthy to note that consumption at highest levels of wealth and indebtedness is identical across both states of the economy, implying the insignificance of idiosyncratic shocks on wealthy households. The idiosyncratic shock only comes into play when households are liquidity constrained.

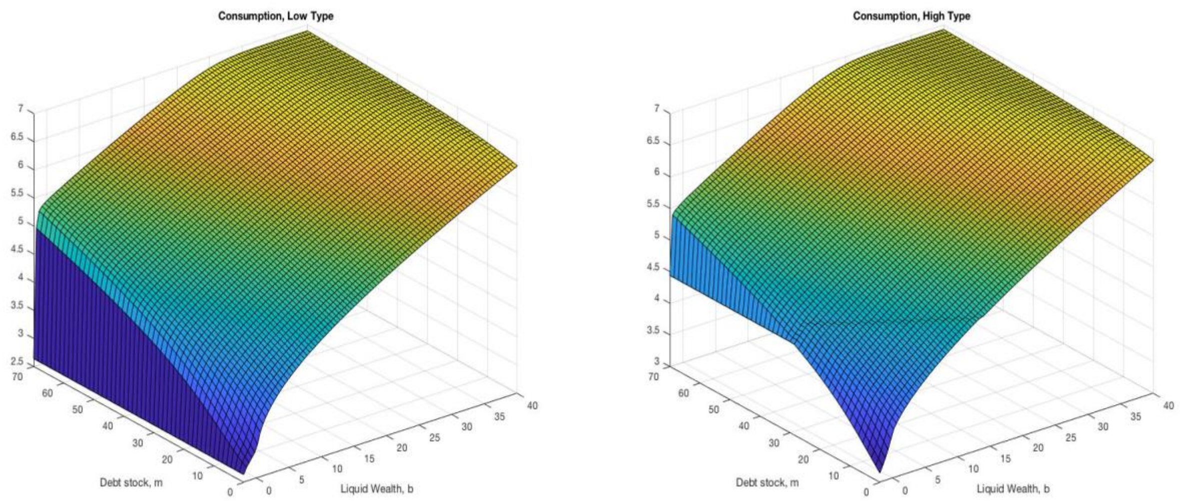

Figure 1: Households' consumption in low and high types

Figure 2 plots the optimal liquid savings policies of households under the current settings. We can observe that there is rich heterogeneity across households in terms of savings. Household savings in liquid assets increases with the amount of mortgage debt. This result is viable in the sense that those who have higher income or liquid wealth stock will be able to get larger amounts of subsidized loans. Aside from the rich heterogeneity in savings behavior, the shape of the savings policy function is consistent with that of heterogeneous agent models with a one-asset structure, with higher savings incentives at the borrowing constraint for the low productivity economy due to precautionary savings motives.

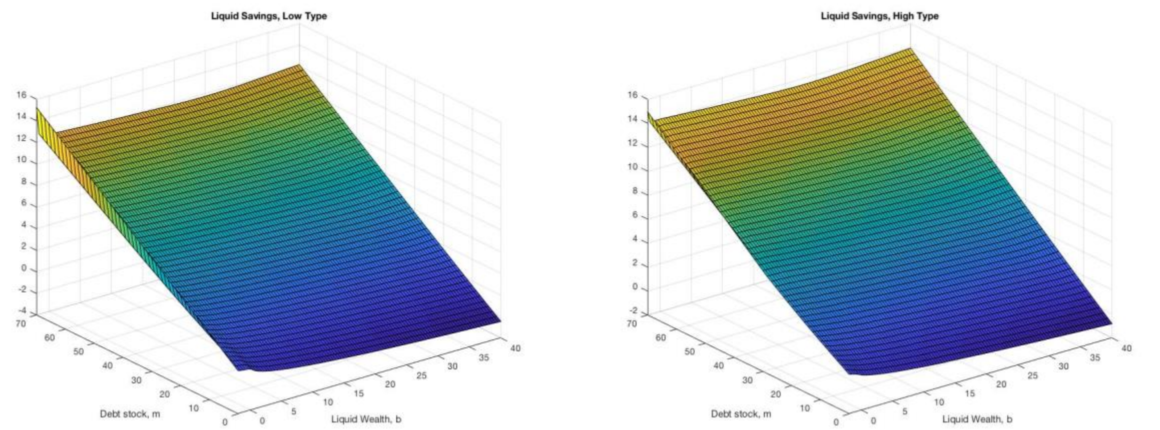

Figure 2: Household savings in low and high types

7. Discussion

In this paper, I have introduced and solved a heterogeneous agent model with liquid wealth and mortgage debt in a New Keynesian framework. Through the solution procedure, I have proven that the general equilibrium for this type of model cannot be solved using standard analytical methods. Therefore, I propose the finite difference upwind scheme method, and apply it for the treatment of this model. This solution method is effective and robust, and can be used to solve a wide range of heterogeneous agent models.

Furthermore, this model was structured in a way that can be a useful starting point for studying aspects of the economy related to heterogeneities in household income, wealth and asset structure, and how they can affect the aggregate economy, and vice versa. Additionally, and more specifically, the proposed model can be used for further studies on the effects of (subsidized or not) mortgage loans on a wide range of macroeconomic variables.

From the experimental exercise, we observe that consumption smoothing and precautionary savings motives still persist in this setting, albeit at a lower level. Heterogeneity in savings behavior is high in both states of the economy. We also observe that the effect of idiosyncratic shock is significant for cash-constrained households.

A key limitation of this model is the lack of standardized computational solution methods, especially those associated with connecting household heterogeneity with the rest of the economy. Further research in this area is much needed and anticipated to open up an extensive array of research possibilities related to macroeconomic policy and welfare analysis.

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APPENDIX I

Finite Difference Method

The upwind finite difference scheme is used to solve the HJB equation. I split the drift of $b, wz_k + r^b b - d - \kappa(d,m) - c$ into two parts: $s^c = wz_k - \tilde{T} + r^b b - c$ and $s^d = -d - \kappa(d,m)$, and use the upwind scheme on them separately.

To start the discretization procedure, denote grid points $b_i, i = 1, \dots, I, m_j, j = 1, \dots, J, z_k, k = 1, \dots, K$. Thus, we have:

$$V_{i,j,k} = V(b_i, m_j, z_k)$$

Denote $\Delta b_i^+ = b_{i+1} - b_i$ and $\Delta b_i^- = b_i - b_{i-1}$ and so on, and approximate derivatives for b and m with either a forward or a backward difference approximation:

$$V_b(b_i, m_j, z_k) \approx V_{b,i,j,k}^F = \frac{V_{i+1,j,k} - V_{i,j,k}}{\Delta b_i^+}$$

$$V_b(b_i, m_j, z_k) \approx V_{b,i,j,k}^B = \frac{V_{i,j,k} - V_{i-1,j,k}}{\Delta b_i^-}$$

Repeat the same process for V_m, and derive the discretized version of the HJB equation.

$$\begin{aligned}\frac{V_{i,j,k}^{n+1} - V_{i,j,k}^n}{\Delta} + \rho V_{i,j,k}^{n+1} \\ &= u(c_{i,j,k}^n) + V_{b,i,j,k}^{n+1} s_{i,j,k}^{b,n} + V_{m,i,j,k}^{n+1}(-r^m m_j - d_{i,j,k}^n) \\ &+ \sum_{k' \neq k}^{K} \lambda_{k,k'}(V_{i,j,k}^{n+1} - V_{i,j,k}^{n+1})\end{aligned} \quad (102)$$

$$s_{i,j,k}^n = wz_k - \tilde{T} + r_b b_i - d_{i,j,k}^n - \kappa(d_{i,j,k}^n, m_j) - c_{i,j,k}^n \quad (103)$$

$$u'(c_{i,j,k}^n) = V_{b,i,j,k}^n \quad (104)$$

$$V^n_{b,i,j,k}\left(-1-\kappa_d\left(d^n_{i,j,k},m_j\right)\right)=V^n_{m,i,j,k} \tag{105}$$

The choice of backward or forward difference is decided by the following "rule": use a forward difference whenever the drift of a state variable is positive, and vice versa. Now define:

$$s^{c,B}_{i,j,k}=wz_k-\tilde{T}+r^b b_i-c^{B,n}_{i,j,k}$$
$$s^{c,F}_{i,j,k}=wz_k-\tilde{T}+r^b b_i-c^{F,n}_{i,j,k}$$

where $c^{B,n}_{i,j,k}$ is optimal consumption decision implicitly calculated using backward difference of b_i, and $c^{F,n}_{i,j,k}$ is optimal consumption decision calculated using forward difference of b_i.

Now, the following approximation must be carried out:

$$V\left(b_i,m_j,z_k\right)s^c(\left(b_i,m_j,z_k\right)\approx V^{B,n+1}_{b,i,j,k}\left(s^{c,B}_{i,j,k}\right)^-+V^{F,n+1}_{b,i,j,k}\left(s^{c,F}_{i,j,k}\right)^+$$

Define $d^{BB}_{i.j.k},d^{BF}_{i.j.k},d^{FB}_{i.j.k},d^{FF}_{i.j.k}$ as optimal repayments calculated using the forward/backward difference approximation with respect to b, $V^{F/B}_{b,i,j,k}$, and the forward/backward difference approximation with respect to m, $V^{F/B}_{m,i,j,k}$.

Furthermore, substituting these definitions into the overall drifts of d and s, we get:

$$d^B_{i,j,k}=\left(d^{BF}_{i,j,k}\right)^++\left(d^{BB}_{i,j,k}\right)^- \tag{106}$$
$$d^F_{i,j,k}=\left(d^{FF}_{i,j,k}\right)^++\left(d^{FB}_{i,j,k}\right)^- \tag{107}$$
$$s^{d,B}_{i,j,k}=-d^{B,n}_{i,j,k}-\kappa\left(d^{B,n}_{i,j,k},m_j\right) \tag{108}$$
$$s^{d,F}_{i,j,k}=-d^{F,n}_{i,j,k}-\kappa\left(d^{F,n}_{i,j,k},m_j\right) \tag{109}$$
$$d_{i,j,k}=d^B_{i,j,k}\mathbf{1}_{\{s^{d,B}_{i,j,k<0}\}}+d^F_{i,j,k}\mathbf{1}_{\{s^{d,B}_{i,j,k>0}\}} \tag{110}$$

Substitute equations (106) – (110) into the HJB equation to obtain:

$$
\begin{aligned}
&\frac{V_{i,j,k}^{n+1}-V_{i,j,k}^{n}}{\Delta}+\rho V_{i,j,k}^{n+1} \\
&\qquad = u\left(c_{i,j,k}^{n}\right)+V_{b,i,j,k}^{B,n+1}\left(s_{i,j,k}^{c,B}\right)^{-}+V_{b,i,j,k}^{F,n+1}\left(s_{i,j,k}^{c,F}\right)^{+}+V_{b,i,j,k}^{B,n+1}\left(s_{i,j,k}^{d,B}\right)^{-} \\
&\qquad + V_{b,i,j,k}^{F,n+1}\left(s_{i,j,k}^{d,F}\right)^{+}+V_{m,i,j,k}^{B,n+1}d_{,i,j,k}^{-}+V_{m,i,j,k}^{B,n+1}(-d_{,i,j,k}^{+}-r^{m}m_{j}) \\
&\qquad + \sum_{k'\neq k}^{K}\lambda_{k,k'}((V_{i,j,k}^{n+1}-V_{i,j,k}^{n+1})
\end{aligned}
\tag{111}
$$